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NUMERICAL METHODS IN WEATHER PREDICTION: II. SMOOTHING AND FILTERING*

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ABSTRACT

A method is developed for the design of finite-difference smoothing and filtering operators which meet predetermined specifications, and which are applicable to automatic computing machinery. The general technique is to build complicated operators from the simplest types. The necessity for smoothing predicted fields of stream functions before inverting the balance equation for heights of isobaric surfaces is brought out.

1. INTRODUCTION

Numerical weather prediction, making use of finite differences and digital computers, has invariably led to amplification of high frequency components in the final product—amplification beyond physical reality. An excess of short-wavelength components detracts from the appearance of the product, is annoying to analysts, and can be downright misleading to the uninitiated. A method of constructing filtering, or "smoothing", operators was devised by the author [1], which have been successfully employed in operational practice to eliminate short-wavelength components from fields of meteorological variables. In one aspect of operational numerical weather prediction, it has proven necessary in the interests of accuracy to filter out the short-wavelength components. Section 6 will deal with this.

2. THE SMOOTHING ELEMENT

First of all, let a smoothing element be defined. The smoothing element will be the building block of more complicated smoothing operators. We shall take as the smoothing element the simplest of one-dimensional symmetrical centered finite difference operators which does not affect the mean value of a field of infinite extent, namely,

$$\bar{z}_i=\mu z_i+\frac{1}{2}\,(1-\mu)(z_{i-1}+z_{i+1}) \tag{1}$$

where z is the field to be smoothed. The subscripts refer to points equally spaced in the independent variable, x, and consecutively numbered with increasing x. For imminent conceptual convenience, we will rewrite equation (1).

$$\bar{z}_i=z_i+\frac{1}{2}\,\nu(z_{i-1}-2z_i+z_{i+1}) \tag{2}$$

where

$$\nu=1-\mu.$$

The parameter ν which is twice the weight given the two outer points, will be called the smoothing element index, since it completely defines a given operator of form (1) or (2).

It will be convenient to think of the dependent variable, z, within the region of interest as consisting of the sum of trigonometric (cosine, say) functions of varying amplitudes, phases, and wave numbers. According to this concept, the wavelengths need not be restricted to multiples of the finite-difference increment in x, nor need the number of waves for a given component within the region

*For Part I see *Monthly Weather Review*, October 1957, pp. 329–332. The third, and last, part in this series will appear in a future issue.

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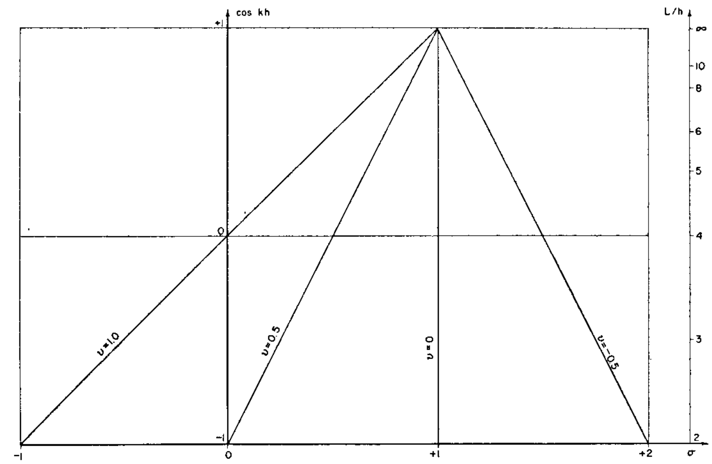

FIGURE 1.—The field of smoothing element index in (σ, cos kh)-space. σ is the ratio of smoothed to unsmoothed amplitude, k is the wave number, h is mesh length, L is wavelength.

of interest be integral. Adopting this concept, we will investigate the effect of a smoothing element on individual cosine components. For example, consider the component

$$z_i = C + A \cos k(x_i - \bar{x})$$

where $\bar{x}$ is an arbitrary constant related to the phase and k is the wave number; i. e., $k = 2\pi/L$, where L is the wavelength of the component. Trigonometric identities yield

$$z_{i+1} = C + A \cos k(x_{i+1} - \bar{x}) = C + A \cos k(x_i - \bar{x} + h)$$
$$= C + A \cos kh \cos k(x_i - \bar{x}) - A \sin kh \sin k(x_i - \bar{x})$$
$$z_{i-1} = C + A \cos k(x_{i-1} - \bar{x}) = C + A \cos k(x_i - \bar{x} - h)$$
$$= C + A \cos kh \cos k(x_i - \bar{x}) + A \sin kh \sin k(x_i - \bar{x})$$

where h is the length of the finite difference increment in x. If these identities are substituted into equation (2), we have, after some rearrangements,

$$\bar{z}_i = C + [1 - \nu(1 - \cos kh)] A \cos k(x_i - \bar{x})$$

Thus, the smoothing element (2) changes neither the wave number nor the phase, but changes the amplitude of each component by the factor

$$\sigma = \frac{\bar{A}}{A} = 1 - \nu(1 - \cos kh) \tag{3}$$

where A and $\bar{A}$ are the amplitudes of the field before and after smoothing, respectively.

Figure 1 shows the field of ν in (σ, cos kh)-space. It is to be noted that an index of zero does not change the field, and negative indices lead to an increase in amplitude of all components. Positive indices lead to an algebraic decrease in amplitude of all components, although indices greater than unity lead to amplifying oscillations. We thus have a conceptual basis in the sign of the smoothing element index for "zero" smoothing, "negative" smoothing, and "positive" smoothing. It is to be noted further that a smoothing element is not highly selective, so would be a poor filtering operator by itself. For example, if we were to filter out of a field components of wavelength $2h$ (cos $kh = -1$) by means of one smoothing element, components of length $10h$ (cos $kh = 0.8$) would be reduced by as much as 10 percent (see the line corresponding to $\nu = 0.5$).

3. THE DESIGN OF MULTI-ELEMENT OPERATORS

In order to improve on the selectivity of the 3-point smoothing element, a smoothing operator must be invented which involves more points. The problem in the design of such an operator is to fit it to stated specifications. The approach to this problem set forth in this article is based on the use of more than one smoothing element (2). Successive application of several smoothing elements, with indices $\nu_0, \nu_1, \nu_2, \nu_3, \ldots, \nu_n$ results in the final ratio of smoothed amplitude to unsmoothed amplitude of

$$\Sigma = \sigma_0 \sigma_1 \sigma_2 \sigma_3 \ldots \sigma_n = \prod_{m=0}^{m=n} [1 - \nu_m(1 - \cos kh)] \tag{4}$$

according to equation (3). Equation (4) is a polynomial in (cos kh), with $n+1$ degrees of freedom, represented by the arbitrary constants $\nu_0, \nu_1, \nu_2, \ldots \nu_n$. In principle, one could specify a single-valued curve of Σ against (cos kh) and express it in terms of a product of factors of form (3). One would then know precisely how to accomplish the smoothing desired. In practice, however, this would present a formidable task and furthermore, one is not usually concerned with a precise distribution of Σ in (cos kh). A great deal of improvement, in terms of the desired smoothing end-product, is obtained by combining only two smoothing elements. At the Joint Numerical Weather Prediction (JNWP) Unit, we have not found it necessary as yet to go beyond a combination of three smoothing elements. Our most frequently used multi-element operator will be described in section 5.

4. SMOOTHING IN TWO DIMENSIONS

Extension of the theory to two dimensions may be accomplished in two ways. First, one may smooth in each dimension, independently of the other dimension. It can be shown that the final result is independent of the dimension in which one first smooths, and is also independent of the order in which one applies the smoothing elements. Adopting the view that such an extension of a smoothing element to two dimensions is really the application of two smoothing elements, one in each dimension, the two elements may be combined into a single 9-point operator.

$$\bar{z}_0 = z_0 + \frac{1}{2}\nu(1-\nu)(z_2 + z_4 + z_6 + z_8 - 4z_0) + \frac{1}{4}\nu^2(z_1 + z_3 + z_5 + z_7 - 4z_0) \tag{5}$$

The subscripts refer to mesh points in figure 2, ν being the index of the two smoothing elements, one applied in each dimension.

If, for convenience, we consider the function $z\ (x,\ y)$ to be composed of the sum of two-dimensional trigonometric components of form

$$z(x,y)=C+A\cos r(x-\overline{x})\cos s(y-\overline{y}), \qquad (6)$$

the ratio of smoothed to unsmoothed amplitudes is

$$\sigma=\frac{\overline{A}}{A}=[1-\nu(1-\cos rh)][1-\nu(1-\cos sh)].$$

Since each of the two factors is of the form of the right hand side of equation (3), each factor may be evaluated by means of figure 1.

The second way of extending the theory to two dimensions is through the 5-point operator,

$$\overline{z}_0=z_0+\frac{1}{4}\nu(z_2+z_4+z_6+z_8-4z_0) \qquad (7)$$

An analysis of the effect of such an operator on a component (6) reveals that the ratio of the smoothed to unsmoothed amplitudes is

$$\sigma=\frac{\overline{A}}{A}=1-\nu\left[1-\frac{1}{2}(\cos rh+\cos sh)\right]$$

Thus, if we were to replace the ordinate ($\cos kh$) in figure 1 by $\frac{1}{2}(\cos rh+\cos sh)$, the figure would then apply to the 5-point operator (7).

If the component (6) represented a "wiggle" in one dimension only, (e. g., $r=0$, $s=2\pi/2h$) the 9-point operator (5) with $\nu=0.5$ would eliminate it, treating it as a one-dimensional element would treat it. The 5-point operator (7), on the other hand, would reduce it by only one-half treating it as a one-dimensional operator would treat a component of wave number $k=2\pi/4h$. Because of this characteristic of the 5-point operator, we have found little use for it. We use almost exclusively combinations of 9-point operators.

The extensions to two dimensions described here have obvious analogues in extensions to spaces of more than two dimensions.

5. COMPLEX SMOOTHING ELEMENT INDICES

There is nothing in the theory which rules out complex indices. A combination of two smoothing elements is equivalent to a single 5-point one-dimensional smoothing operator. If the two smoothing elements have conjugate complex indices, the weights at the five points will be real. Conversely, any 5-point one-dimensional operator is equivalent to a combination of two smoothing elements. Acceptance of complex indices into the theory merely allows this converse to be perfectly general.

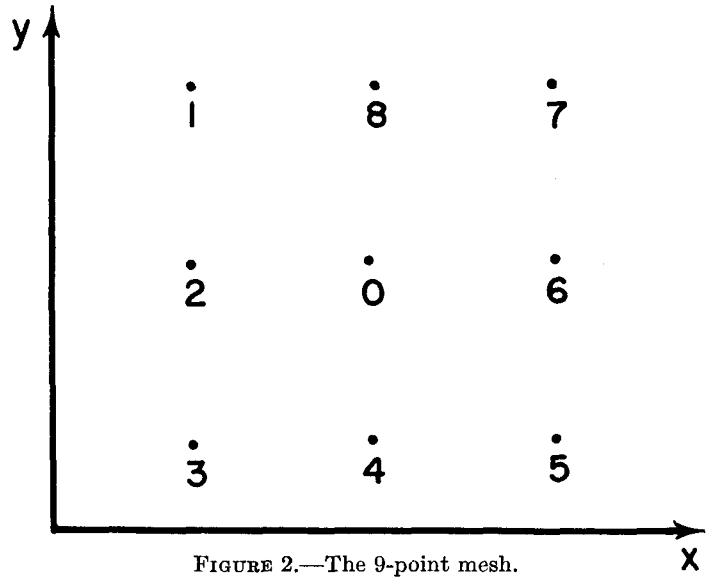

FIGURE 2.—The 9-point mesh.

The multi-element operator in most frequent use at the JNWP Unit consists of an element with a real index, and two elements with conjugate complex indices; i. e.,

$$\begin{aligned}\nu_0&=0.49965\\ \nu_1&=-0.22227+0.64240i\\ \nu_2&=-0.22227-0.64240i\end{aligned} \qquad (8)$$

A 3-element operator allows the specification of three characteristics of the curve of Σ against ($\cos kh$). In practice, we have specifications which are not precisely stated, so cannot be handled easily by rigorous mathematical methods.

For example, we want our operational smoothing operator to severely suppress the short waves while retaining essentially unchanged the longer waves. By combining algebraic and graphical techniques, we have arrived at the three elements whose indices (8) are recorded above, and whose effects are displayed graphically in figure 3. Figure 3 shows the result of both one pass and thirty-two passes, the latter to bring out clearly the form for the longer wavelengths.

Complex indices always appear in conjugate pairs, otherwise the multi-element operator would result in imaginary components. Conjugate complex pairs of 9-point operators require smoothing on the boundary for the same reason. We apply the corresponding one-dimensional elements to the boundary, so that in the case of a rectangular grid only the four corner points remain unchanged.

When one comes to programming for automatic computing machinery, the question arises as to whether to perform one scan for each element in a multi-element operator, or to combine the elements into one large smoothing operator. In the case of the operator cited above, if the elements were combined into one large operator, it would be applied to a 7 x 7 mesh of 49 points. More importantly, there would be 10 classes of central

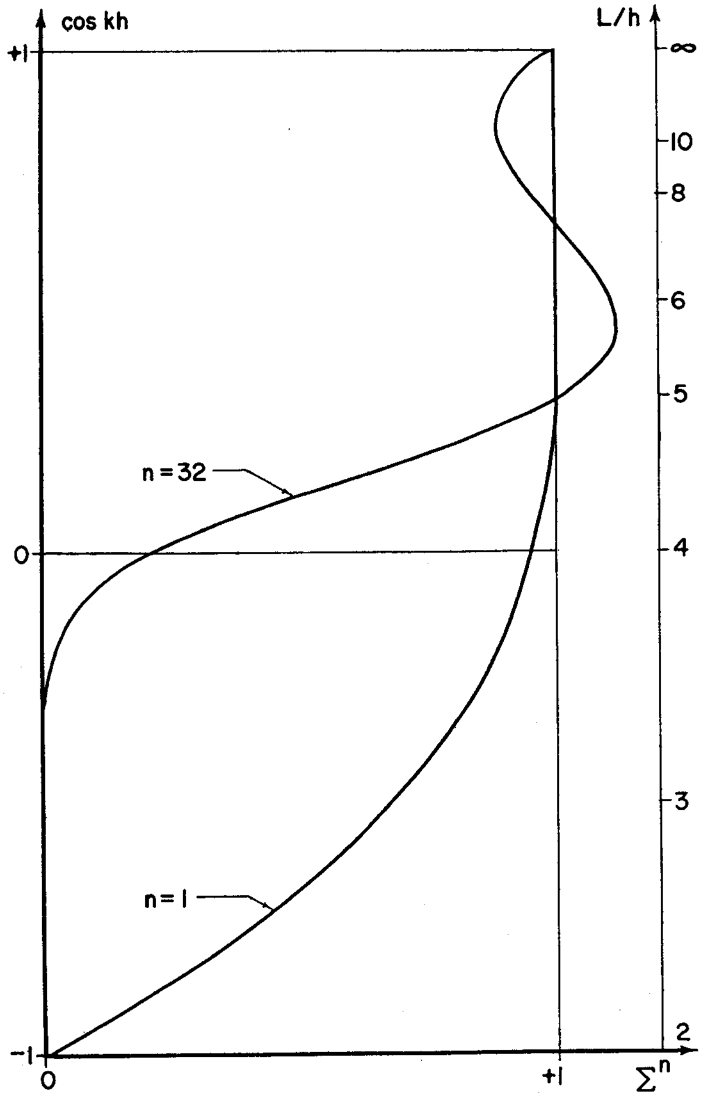

FIGURE 3.—The ratio, Σ, of smoothed to unsmoothed amplitude after N smoothings with the multi-element operator whose smoothing element indices are $\nu_0=0.49965$, $\nu_1=-0.22227+0.64240i$, $\nu_2=-0.22227-0.64240i$.

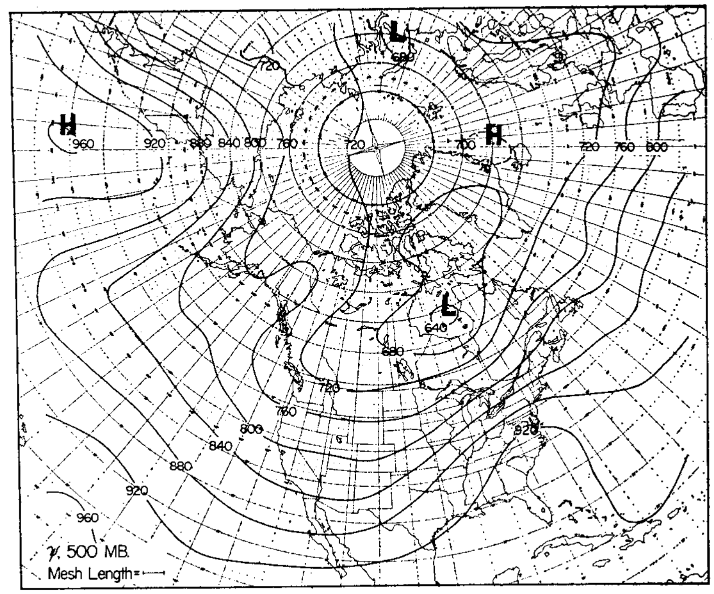

FIGURE 4.—The predicted field of $\bar{f}g^{-1}\psi$ 72 hours after 0300 GMT, April 26, 1956, smoothed 3 times with the multi-element operator (8). Contours are labeled in tens of feet.

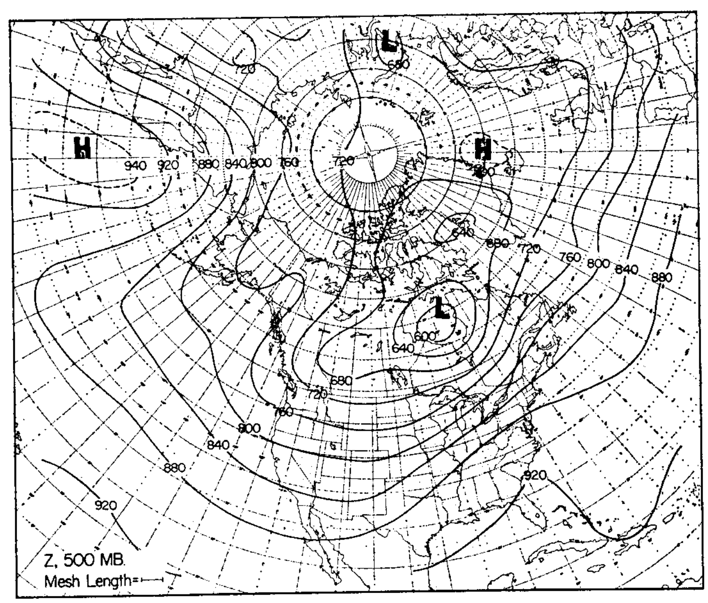

FIGURE 5.—The predicted field of z 72 hours after 0300 GMT, April 26, 1956, inverted from the balance equation and the smoothed field of ψ depicted in figure 4. Contours are labeled in tens of feet.

points, each requiring different treatment. For this reason we believe the advantage lies in performing one scan for each element, the one utility program then handling all multi-element operators for a given grid.

Examples of the application of the smoothing operator represented by the indices (8) will be given in the next section.

6. INVERSION OF THE BALANCE EQUATION FOR THE GEOPOTENTIAL

The one area in which we have found smoothing mandatory in the interest of accuracy is in connection with the inversion of the balance equation (Shuman [2]),

$$\frac{1}{2}(\psi_{xx}+\psi_{yy}+f)^2-\frac{1}{2}(\psi_{xx}-\psi_{yy})^2-2\psi_{xy}^2+\psi_x f_x+\psi_y f_y-\left(gz_{xx}+gz_{yy}+\frac{1}{2}f^2\right)=0 \quad (9)$$

where z is the height of the 500-mb. contour, ψ is the stream function for the winds at 500 mb., g is gravitational acceleration, f is the Coriolis parameter. Figure 4 shows a 72-hr. barotropic prediction of the ψ-field made with wind fields which satisfy the balance equation. Figure 4 is the predicted ψ-field after being smoothed three times with the operator (8). The ψ-field before smoothing is not shown, but the differences between the smoothed and nonsmoothed fields are portrayed by the lighter curves of figure 6. Only the zero isopleth is shown. The sense (plus or minus) of the differences is not indicated, since it

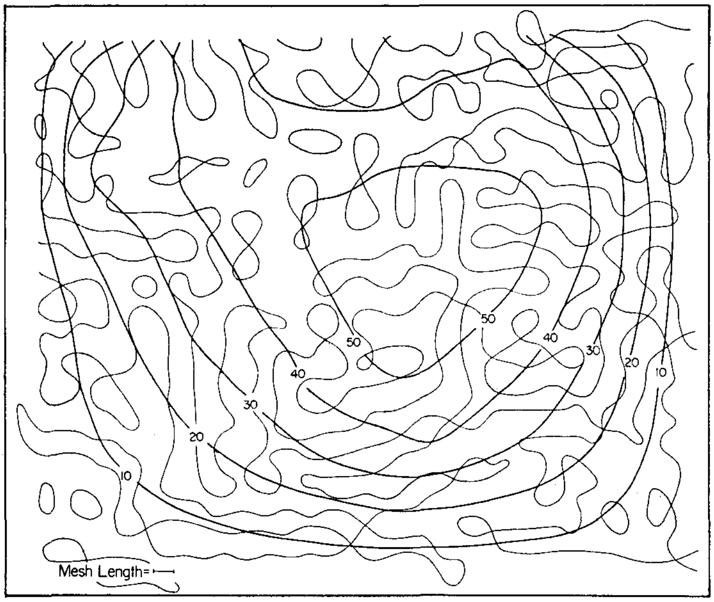

FIGURE 6.—The light curves are the zero isopleth depicting the choppy field of the differences between the ψ-field of figure 4 before and after smoothing. The heavy curves depict the large-scale effect of smoothing the ψ-field, on the z-field implied by the balance equation. The smoothed ψ-field implies a z-field generally higher than the unsmoothed ψ-field. Contours are labeled in tens of feet.

is of no particular interest. The important characteristics of the difference field to be noted are its raggedness, or choppiness, and its lack of consistency in the sense that its mean value is not different from zero.

Figure 5 is the solution of the balance equation (9) for z, with ψ taken as the smoothed field of ψ (fig. 4). The balance equation was also inverted for z with ψ taken as the ψ-field before smoothing. The latter result is not shown, but the differences between the two z-fields are shown by the smooth heavier set of curves in figure 6.

Figure 6 shows that a high-frequency change in the ψ-field implies a very low-frequency change in the z-field through the balance equation (at least through our finite-difference version of it). This result must be due to the non-linearity of the balance equation in ψ.

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2. F. G. Shuman, "Numerical Methods in Weather Prediction: I. The Balance Equation," *Monthly Weather Review*, vol. 85, No. 9, Oct. 1957, pp. 329–332.